Convexification of Bilinear Terms over Network Polytopes

MIP Workshop 2024

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Joint work with:

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Outline of the talk

Part I: Introduction

Part II: Aggregation Method

Part III: Convexification for $m = 1$

Part IV: Convexification for $m > 1$

Part V: Computational Experiments

Problem definition

Consider

$$\mathcal{S} = \{(\boldsymbol{x};\boldsymbol{y};\boldsymbol{z}) \in \Sigma \times \Delta_m \times \mathbb{R}^{\kappa} \,|\, x_i y_j = z_{ij}, \;\; \forall (i,j) \in N \times M\},$$

where

- $\Sigma = \{\boldsymbol{x} \in \mathbb{R}^n \,|\, E\boldsymbol{x} \geq \boldsymbol{f},\, \boldsymbol{0} \leq \boldsymbol{x} \leq \boldsymbol{u}\}$ is a network polytope described by the flow-balance and arc capacity constraints
- $\Delta_m = \left\{\boldsymbol{y} \in \mathbb{R}^m_+ \,\middle|\, \boldsymbol{1}^\intercal \boldsymbol{y} \leq 1\right\}$ is a simplex
  - Naturally imposed: SOS1
  - Reformulated: $\mathcal{V}$-representation of polytopes

Applications

Structure of set $\mathcal{S}$ appears in various optimization models in different application areas such as:

- Fixed-charge network flow problems
- Transportation problem with conflicts
- Bilevel network flow problems
- Network interdiction problems
- Optimization via decision diagrams
- ...

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Studying convexification of $\mathcal{S}$ can lead to cutting planes that can tighten existing relaxations and improve dual bounds.

Motivation

McCormick relaxation [McCormick, 1976]

- Replace $z = xy$ with $z \geq 0$, $z \geq x + y - 1$, $z \leq x$, $z \leq y$ for unit box domain on x and y.
- Provides convex hull over box domains [Al-Khayal & Falk, 1983].
- Often leads to weak relaxations when the domain is more general [Luedtke et. al., 2012], [Gupte et. al., 2013].

Disjunctive programming [Balas, 1985] or special structure RLT [Sherali et. al., 1998]

- Describes convex hull in a higher dimension.
- Can be computationally expensive due to large size of extended formulation.
- Uses separation to generate cuts in the original space of variables.
- Does not provide explicit forms of facet-defining inequalities in the original space of variables.

Goal

Design a systematic procedure to obtain an explicit form of the convex hull description in the original space of variables

Outline of the talk

An aggregation procedure

We use a specialized aggregation procedure, called *Extended Cancel & Relax (EC&R)*, to obtain valid inequalities for $\mathrm{conv}(\mathcal{S})$

[Davarnia, Richard, Tawarmanali, 2017]

An aggregation procedure

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[Davarnia, Richard, Tawarmanali, 2017]

Step 1: Assign weights

- Pick a bilinear constraint with weight ± 1 (base constraint)

$$(\pm 1) \times (x_k y_l - z_{kl} = 0)$$

- Pick linear constraints from Σ with the following weights:

$$\beta \begin{pmatrix} y_1 \\ \vdots \\ y_m \\ 1 - \sum_{j \in M} y_j \end{pmatrix} \times \begin{pmatrix} E_{t.}\boldsymbol{x} & \geq f_t \\ x_i & \geq 0 \\ 1 - x_i & \geq 0 \end{pmatrix} \qquad \mathcal{I}$$

An aggregation procedure

Step 2: Aggregate the above weighted inequalities such that at least $|\mathcal{I}|$ bilinear terms cancel

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Step 3: Relax the remaining bilinear term using McCormick bounds or bilinear constraints

- Replace $x_i y_j$ with
  - $u_i y_j$
  - x_i
  - z_{ij}

- Replace $-x_i y_j$ with
  - 0
  - $x_i + u_i y_j - u_i$
  - $-z_{ij}$

EC&R theorem - proof sketch

Theorem

A linear description of $\mathrm{conv}(\mathcal{S})$ is given by:

- the inequalities defining Σ,
- the inequalities defining Δ_m,
- all EC&R inequalities.

EC&R theorem - proof sketch

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- the inequalities defining Δ_m,
- all EC&R inequalities.

Proof sketch:

- Observe that $\mathcal{S}$ is bounded.
- The vertices of $\mathcal{S}$ are such that $y = e_i$ or $y = 0$.
- The restriction of $\mathcal{S}$ to $y = e_i$ ($y = 0$) is polyhedral.
- The convex hull of $\mathcal{S}$ can be obtained in higher dimension using disjunctive programming.
- The rays of the projection cone of the disjunctive programming formulation have structure.
- The components of the rays can be interpreted as "dual" weights on the initial constraints of the system that "cancel" product variables.

Outline of the talk

Level-1 generalization of McCormick

McCormick relaxation:

$$xy = z$$

over

- $x \in [0, u]$
- $y \in [0, 1]$

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Level-1 generalization:

$$x_i y = z_i, \qquad \forall i \in A$$

over

- $\boldsymbol{x} \in \Sigma = \{\boldsymbol{x} \in \mathbb{R}^n \,|\, E\boldsymbol{x} \geq \boldsymbol{f}, \, \boldsymbol{0} \leq \boldsymbol{x} \leq \boldsymbol{u}\}$
- $y \in [0, 1]$

Determining aggregation weights

Proposition

Let $\boldsymbol{a}^\top \boldsymbol{x} + by + \boldsymbol{c}^\top \boldsymbol{z} \geq d$ be a non-trivial facet-defining inequality of $\mathrm{conv}(\mathcal{S})$ that is obtained from the EC&R procedure. Then, the weights β of all network constraints used in the aggregation are equal to 1.

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Proof sketch:

- The projection problem for the disjunctive programming formulation has the following form

$$\left[\begin{array}{c|c||c|c||c|c} E^\intercal & -E^\intercal & \pm I & \pm I & I & -I \end{array}\right]\boldsymbol{\pi} = \pm e^k.$$

- The coefficient matrix is TU
- The RHS is a unit vector
- The weights $\boldsymbol{\pi}$ are non-negative

Graphical structure of EC&R inequalities

Theorem

Consider

- $\mathcal{S}$ with $m = 1$ defined over a network $G = (V, A)$,
- $\boldsymbol{a}^\top \boldsymbol{x} + by + \boldsymbol{c}^\top \boldsymbol{z} \geq d$ to be a non-trivial facet-defining inequality of $\text{conv}(\mathcal{S})$ that is obtained from the EC&R procedure.
- $\mathcal{I}$ to be the set of flow-balance constraints used in the aggregation.

Then,

- The nodes corresponding to constraints in $\mathcal{I}$ form a tree in G.

Graphical structure of EC&R inequalities

Proof sketch (part I):

- From previous proposition, to cancel a bilinear term $x_i y$ for some $i \in A$, we need to use 2 flow-balance constraints at nodes $t(i)$ and $h(i)$

$$1 \times y \times \left(\sum_{j \in \delta^+(t(i)) \setminus \{i\}} x_j - \sum_{j \in \delta^-(t(i))} x_j + x_i \geq f_{t(i)} \right)$$

$$1 \times y \times \left(\sum_{j \in \delta^+(h(i))} x_j - \sum_{j \in \delta^-(h(i)) \setminus \{i\}} x_j - x_i \geq f_{h(i)} \right)$$

- The nodes whose flow-balance constraints are used in the aggregation must be adjacent

Graphical structure of EC&R inequalities

Proof sketch (part II):

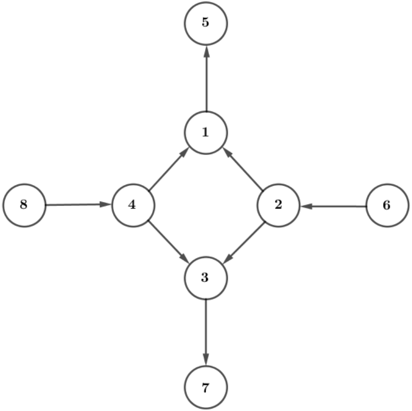
- This subnetwork must be connected.
  - *Contradiction:* for a disconnected subnetwork, the basis has the following form.

$$\left[\begin{array}{cc|cc} \pm E_1 & \pm I_1 & 0 & 0 \\ \hline 0 & 0 & \pm E_2 & \pm I_2 \end{array}\right] \left[\begin{array}{c} \mathbf{1} \\ \hline \mathbf{1} \end{array}\right] = \left[\begin{array}{c} \pm \boldsymbol{e}^i \\ \hline \mathbf{0} \end{array}\right].$$

  - Columns in second part are linearly dependent.

Example

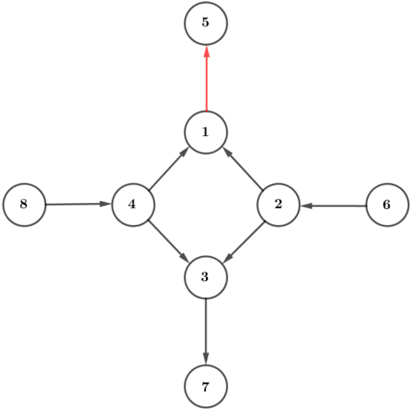
Consider set $\mathcal{S}$ defined over the following network. Assume that we are interested in finding EC&R inequalities with base constraint $-x_{1,5}y + z_{1,5} = 0$.



Example

Consider set $\mathcal{S}$ defined over the following network. Assume that we are interested in finding EC&R inequalities with base constraint $-x_{1,5}y + z_{1,5} = 0$.

$$1 \times (-x_{1,5}y + z_{1,5} = 0)$$



Example

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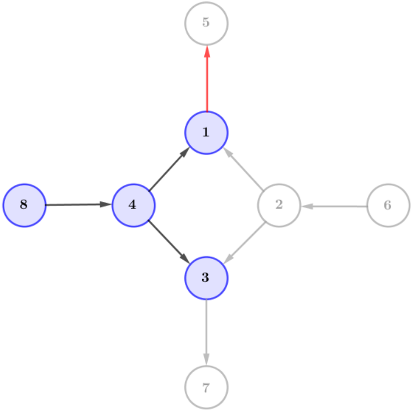
$$1 \times (-x_{1,5}y + z_{1,5} = 0)$$
$$y \times (-x_{4,1} - x_{2,1} + x_{1,5} \geq f_1)$$
$$y \times (x_{4,1} + x_{4,3} - x_{8,4} \geq f_4)$$
$$1 - y \times (-x_{8,4} \geq -f_8)$$
$$y \times (x_{3,7} - x_{4,3} - x_{2,3} \geq f_3)$$



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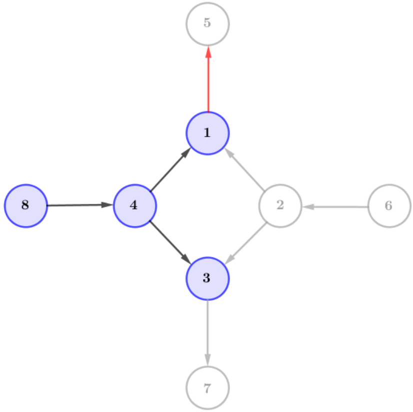
y \times (x_{4,1} + x_{4,3} - x_{8,4} \geq f_4)\\
1 - y \times (-x_{8,4} \geq -f_8)\\
y \times (x_{3,7} - x_{4,3} - x_{2,3} \geq f_3)\\
\hline
-x_{4,1}y - x_{2,1}y + x_{3,7}y - x_{2,3}y\\
-x_{8,4} - (f_1 + f_4 + f_8 + f_3)y\\
+z_{1,5} + f_8 + 0x_{1,5}y\\
+0x_{8,4}y + 0x_{4,1}y + 0x_{4,3}y \geq 0
\end{aligned}
$$



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Level-2 generalization of McCormick

McCormick relaxation:

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Level-2 generalization:

$$x_i y_j = z_{ij}, \qquad \forall i \in A, j \in M$$

over

- $\boldsymbol{x} \in \Sigma = \{\boldsymbol{x} \in \mathbb{R}^n \,|\, E\boldsymbol{x} \geq \boldsymbol{f}, \, \boldsymbol{0} \leq \boldsymbol{x} \leq \boldsymbol{u}\}$
- $\boldsymbol{y} \in \Delta_m = \left\{\boldsymbol{y} \in \mathbb{R}^m_+ \,\middle|\, \boldsymbol{1}^\intercal \boldsymbol{y} \leq 1\right\}$

A more complicated structure

The coefficient matrix in the projection problem for the disjunctive programming formulation has the following form

$$\left[\begin{array}{c|c|c|c||c||c||c|c|c|c}
E^\intercal & \mathbf{0} & \dots & \mathbf{0} & -E^\intercal & \pm I & \pm I & \mathbf{0} & \dots & \mathbf{0} \\ \hline
\mathbf{0} & E^\intercal & \dots & \mathbf{0} & -E^\intercal & \pm I & \mathbf{0} & \pm I & \dots & \mathbf{0} \\ \hline
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \hline
\mathbf{0} & \mathbf{0} & \dots & E^\intercal & -E^\intercal & \pm I & \mathbf{0} & \mathbf{0} & \dots & \pm I
\end{array}\right]$$

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E^\intercal & \mathbf{0} & \dots & \mathbf{0} & -E^\intercal & \pm I & \pm I & \mathbf{0} & \dots & \mathbf{0} \\ \hline
\mathbf{0} & E^\intercal & \dots & \mathbf{0} & -E^\intercal & \pm I & \mathbf{0} & \pm I & \dots & \mathbf{0} \\ \hline
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \hline
\mathbf{0} & \mathbf{0} & \dots & E^\intercal & -E^\intercal & \pm I & \mathbf{0} & \mathbf{0} & \dots & \pm I
\end{array}\right]$$

- The TU property does not hold.
- The aggregation weights for facet-defining EC&R inequalities are not necessarily 1.

A class of facet-defining inequalities

Consider the class of non-trivial facet-defining EC&R inequalities of $\text{conv}(S)$ with pairwise cancellation property.

Definition

An EC&R inequality has pairwise cancellation property if each cancellation of bilinear terms is obtained by aggregation two constraints.

Determining aggregation weights

Proposition

Let $\boldsymbol{a}^\top \boldsymbol{x} + by + \boldsymbol{c}^\top \boldsymbol{z} \geq d$ be a non-trivial facet-defining inequality of $\mathrm{conv}(\mathcal{S})$ that is obtained from the EC&R procedure through pairwise cancellation. Then, the weights β of all network constraints used in the aggregation are equal to 1.

Determining aggregation weights

Proposition

Let $\boldsymbol{a}^\top \boldsymbol{x} + by + \boldsymbol{c}^\top \boldsymbol{z} \geq d$ be a non-trivial facet-defining inequality of $\mathrm{conv}(\mathcal{S})$ that is obtained from the EC&R procedure through pairwise cancellation. Then, the weights β of all network constraints used in the aggregation are equal to 1.

Proof sketch:

- The projection problem for the disjunctive programming formulation has the following form

$$\left[\begin{array}{c|c|c|c} \pm 1 & 0 & \cdots & 0 \\ \hline \{0, \pm 1\} & \pm 1 & \cdots & 0 \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline \{0, \pm 1\} & \{0, \pm 1\} & \cdots & \pm 1 \end{array}\right] \boldsymbol{\pi} = \pm \boldsymbol{e}^1$$

- All (positive) weights must be 1.

New definitions for network structures

Definition

Consider set $\mathcal{S}$ defined over network $\mathrm{G} = (\mathrm{V}, \mathrm{A})$. We define a *parallel network* $\mathrm{G}^j = (\mathrm{V}^j, \mathrm{A}^j)$ for $j \in \{1, \ldots, m\}$ to be a replica of G that represents the multiplication of network constraints with y_j during the aggregation procedure.

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Definition

Consider $\mathcal{S}$ defined over network $\mathrm{G} = (\mathrm{V}, \mathrm{A})$. Let $\widehat{\mathrm{G}}^j = (\widehat{\mathrm{V}}^j, \widehat{\mathrm{A}}^j)$, for $j = 1, 2$, be a subnetwork of parallel network G^j.

- We say that subnetworks $\widehat{\mathrm{G}}^1$ and $\widehat{\mathrm{G}}^2$ are vertically connected through connection node $v \in V$ if the replica of v in parallel network G^j is adjacent to a node of subnetwork $\widehat{\mathrm{G}}^j$ for $j = 1, 2$.
- We say that subnetworks $\widehat{\mathrm{G}}^1$ and $\widehat{\mathrm{G}}^2$ are vertically connected through connection arc $a \in A$ if the replica of a in parallel network G^j is incident to a node of subnetwork $\widehat{\mathrm{G}}^j$ for $j = 1, 2$.

Graphical structure of EC&R inequalities

Theorem

Consider

- $\mathcal{S}$ defined over a network $G = (V, A)$,
- $\boldsymbol{a}^\top \boldsymbol{x} + by + \boldsymbol{c}^\top \boldsymbol{z} \geq d$ to be an EC&R facet-defining inequality of $\mathrm{conv}(\mathcal{S})$ with the pairwise cancellation property.
- $\mathcal{I}^j$ to be the set of flow-balance constraints used in the aggregation multiplied with y_j.
- $\mathcal{J}$ to be the set of flow-balance constraints used in the aggregation multiplied with $1 - \sum_{j=1}^{m} y_j$.
- $\mathcal{K}$ to be the set of variable bound constraints used in the aggregation multiplied with $1 - \sum_{j=1}^{m} y_j$.

Then,

- The nodes corresponding to $\mathcal{I}^j$ form a forest F^j in G^j.
- The forests F^j are vertically connected through the connection nodes in $\mathcal{J}$ and connection arcs in $\mathcal{K}$.

Graphical structure of EC&R inequalities

Proof sketch (part I):

From previous proposition, to cancel a bilinear term $x_i y_j$ for some $i \in A$ and $j \in M$, one of the following cases should occur:

- Two flow-balance constraints at nodes $t(i)$ and $h(i)$ are used in the aggregation:

$$1 \times y_j \times \left(\sum_{j \in \delta^+(t(i)) \setminus \{i\}} x_j - \sum_{j \in \delta^-(t(i))} x_j + x_i \geq f_{t(i)} \right)$$

$$1 \times y_j \times \left(\sum_{j \in \delta^+(h(i))} x_j - \sum_{j \in \delta^-(h(i)) \setminus \{i\}} x_j - x_i \geq f_{h(i)} \right)$$

- The nodes whose flow-balance constraints are used in the aggregation must be adjacent, forming a forest structure in parallel network G^j.

Graphical structure of EC&R inequalities

Proof sketch (part II):

- Two flow-balance constraints at nodes $t(i)$ and $h(i)$ are used in the aggregation:

$$1 \times y_j \times \left(\sum_{j \in \delta^+(t(i)) \setminus \{i\}} x_j - \sum_{j \in \delta^-(t(i))} x_j + x_i \geq f_{t(i)} \right)$$

$$1 \times \left(1 - \sum_{k=1}^{m} y_k \right) \times \left(- \sum_{j \in \delta^+(h(i))} x_j + \sum_{j \in \delta^-(h(i)) \setminus \{i\}} x_j + x_i \geq -f_{h(i)} \right)$$

- One of the nodes whose flow-balance constraint is used in the aggregation must be a connection node.

Graphical structure of EC&R inequalities

Proof sketch (part III):

- A flow-balance constraint at node $t(i)$ and a bound constraint for variable x_i are used in the aggregation:

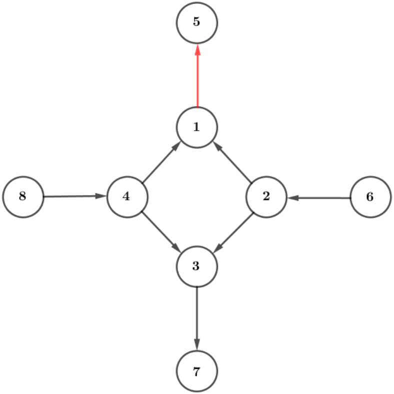
$$1 \times y_j \times \left(\sum_{j \in \delta^+(t(i)) \setminus \{i\}} x_j - \sum_{j \in \delta^-(t(i))} x_j + x_i \geq f_{t(i)} \right)$$

$$1 \times \left(1 - \sum_{k=1}^{m} y_k \right) \times (x_i \geq 0)$$

- The arc whose bound constraint is used in the aggregation must be a connection arc.

Example

Consider set $\mathcal{S}$ with $y_1 + y_2 \leq 1$ defined over the following network. Assume that we are interested in finding EC&R inequalities obtained through pairwise cancellation with base constraint $x_{1,5}y_1 - z_{1,5} = 0$.



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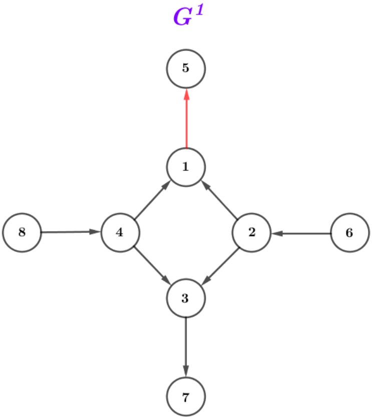

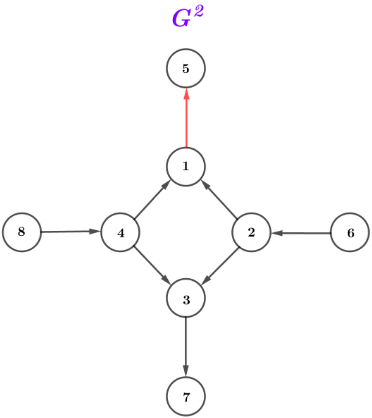

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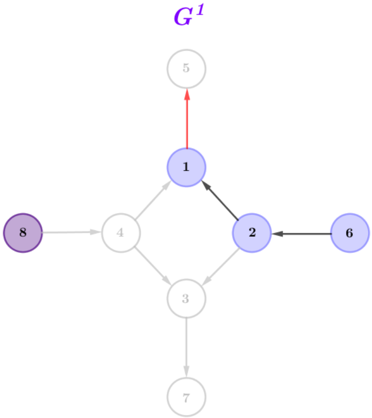

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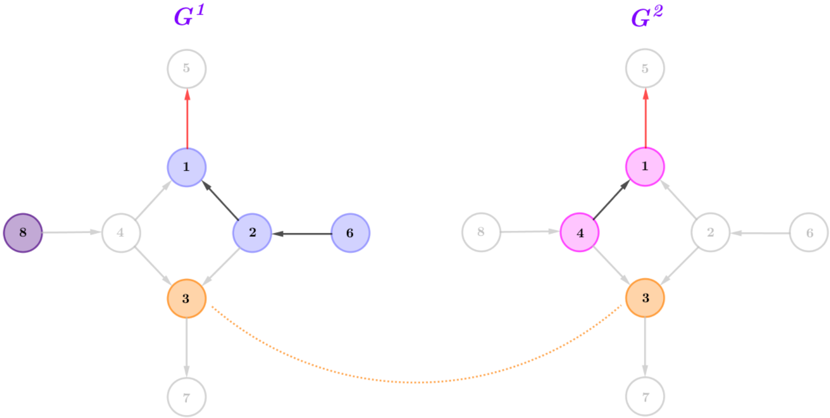

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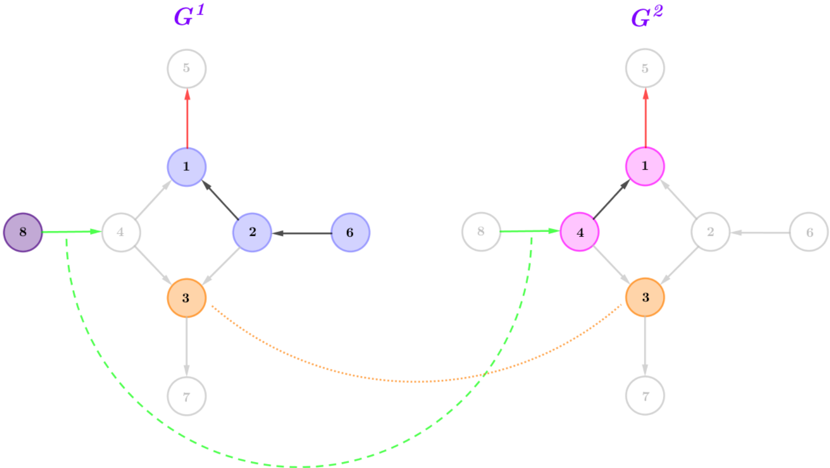
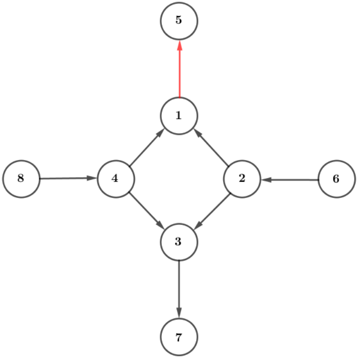


Example

Aggregate the corresponding constraints with appropriate weights.

$$1 \times (x_{1,5} y_1 - z_{1,5} = 0)$$



Example

Aggregate the corresponding constraints with appropriate weights.

$$
\begin{aligned}
1 \times (x_{1,5} y_1 - z_{1,5} = 0) \\
y_1 \times (x_{4,1} + x_{2,1} - x_{1,5} \geq -f_1) \\
y_1 \times (-x_{2,1} - x_{2,3} + x_{6,2} \geq -f_2) \\
y_1 \times (-x_{6,2} \geq -f_6) \\
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\end{aligned}
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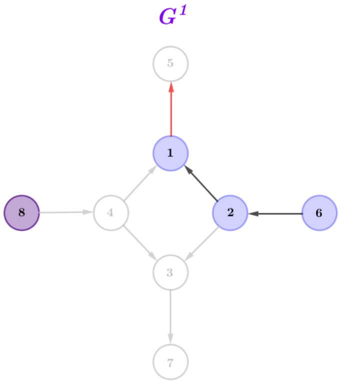

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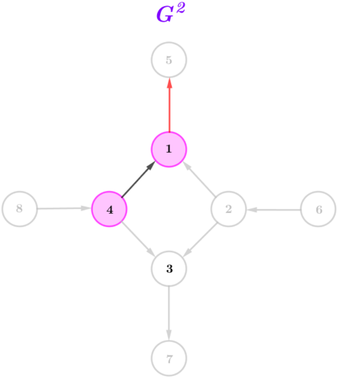
$$y_1 \times (-x_{6,2} \geq -f_6)$$
$$y_1 \times (-x_{8,4} \geq -f_8)$$
$$y_2 \times (x_{4,1} + x_{2,1} - x_{1,5} \geq -f_1)$$
$$y_2 \times (-x_{4,1} - x_{4,3} + x_{8,4} \geq -f_4)$$



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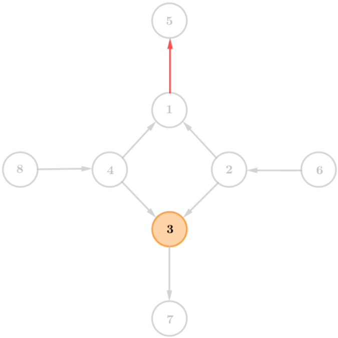
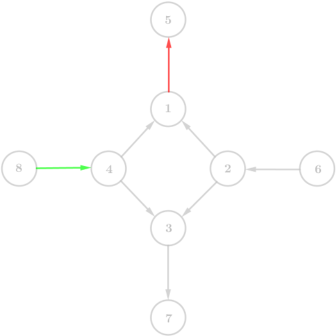
y_1 \times (-x_{8,4} \geq -f_8) \\
y_2 \times (x_{4,1} + x_{2,1} - x_{1,5} \geq -f_1) \\
y_2 \times (-x_{4,1} - x_{4,3} + x_{8,4} \geq -f_4) \\
1 - y_1 - y_2 \times (-x_{2,3} - x_{4,3} + x_{3,7} \geq -f_3)
\end{aligned}
$$



Example

Aggregate the corresponding constraints with appropriate weights.

$$
\begin{aligned}
1 \times (x_{1,5}y_1 - z_{1,5} = 0) \\
y_1 \times (x_{4,1} + x_{2,1} - x_{1,5} \geq -f_1) \\
y_1 \times (-x_{2,1} - x_{2,3} + x_{6,2} \geq -f_2) \\
y_1 \times (-x_{6,2} \geq -f_6) \\
y_1 \times (-x_{8,4} \geq -f_8) \\
y_2 \times (x_{4,1} + x_{2,1} - x_{1,5} \geq -f_1) \\
y_2 \times (-x_{4,1} - x_{4,3} + x_{8,4} \geq -f_4) \\
1 - y_1 - y_2 \times (-x_{2,3} - x_{4,3} + x_{3,7} \geq -f_3) \\
1 - y_1 - y_2 \times (-x_{8,4} \geq -u_{8,4})
\end{aligned}
$$



Example

The aggregated bilinear inequality is

$$
\begin{aligned}
&- y_1x_{4,5} - y_1x_{3,7} + y_1x_{4,3} \\
&\quad - y_2x_{1,5} + y_2x_{4,5} + y_2x_{2,3} - y_2x_{3,7} \\
&\quad + (f_1 + f_2 + f_3 + f_6 + f_8 - u_{8,4})y_1 \\
&\quad + (f_1 + f_3 + f_4 - u_{8,4})y_2 - z_{1,5} \\
&\quad + x_{3,7} - x_{2,3} - x_{4,3} - x_{8,4} - f_3 + u_{8,4} \\
&\quad +0x_{1,5}y_1 + 0x_{2,1}y_1 + 0x_{6,2}y_1 + 0x_{8,4}y_1 \\
&\quad +0x_{2,3}y_1 + 0x_{4,1}y_2 + 0x_{4,3}y_2 + 0x_{8,4}y_2 \geq 0.
\end{aligned}
$$

Relaxing the remaining bilinear terms will yield linear EC&R inequalities.

Outline of the talk

Part I: Introduction

Part II: Aggregation Method

Part III: Convexification for $m = 1$

Part IV: Convexification for $m > 1$

Part V: Computational Experiments

Application I: Fixed-charge network flow problems

$$\min \quad \sum_{i \in S} \sum_{j \in D} \left(\left(c_{ij} + \frac{t_{ij}}{\epsilon_{ij}} \right) x_{ij} + t_{ij} y_{ij} - \frac{t_{ij}}{\epsilon_{ij}} z_{ij} \right)$$

$$x_{ij} y_{ij} = z_{ij}, \qquad \forall i \in S, j \in D$$

$$\sum_{j \in D} x_{ij} \leq s_i, \qquad \forall i \in S$$

$$\sum_{i \in S} x_{ij} \geq d_j, \qquad \forall j \in D$$

$$0 \leq x_{ij} \leq u_{ij}, \qquad \forall i \in S, j \in D$$

$$\sum_{i \in S} \sum_{j \in D} y_{ij} \leq b,$$

$$0 \leq y_{ij} \leq 1, \qquad \forall i \in S, j \in D$$

Application I: Fixed-charge network flow problems

- Used settings from [Rebennack, Nahapetyan, Pardalos, (2009)]
- Number of nodes in bipartite graph: $\{50, 100\}$
- Breakpoint value: $\{0.2, 0.5\}$
- 10 instances for each size category
- Number of y variables: 20% of the arc number
- Used Gurobi to solve different relaxations
- Used EC&R inequalities for up to 3 constraint aggregations
- Used separation to add most violated cuts

Application I: Fixed-charge network flow problems

Node #	Frac.	Solver		Tree EC&R		RLT	
		Opt. Time	Root Gap	Gap	Time	Gap	Time
50	0.2	467.42	0.62	0.75	2.44	0.78	1479.29
50	0.5	186.7	0.63	0.8	2.26	0.84	238.89
100	0.2	$\geq$ 1000	0.41	0.49	8.87	–	$\geq$ 5000
100	0.5	$\geq$ 1000	0.57	0.66	8.89	–	$\geq$ 5000

- The table shows average results over 10 instances for each size category (row)

Application II: Transportation problem with conflicts

$$
\begin{aligned}
\min \quad & \sum_{i\in S}\sum_{j\in D}\left(c_{ij}x_{ij} + \sum_{k\in K} r_{ij}^k z_{ij}^k\right) & \\
& x_{ij}y_k = z_{ij}^k, & \forall i \in S, j \in D, k \in K \\
& \sum_{j\in D} x_{ij} \le s_i, & \forall i \in S \\
& \sum_{i\in S} x_{ij} \ge d_j, & \forall j \in D \\
& 0 \le x_{ij} \le u_{ij}, & \forall i \in S, j \in D \\
& \sum_{k\in L} y_k \le 1, & \forall L \in C \\
& y_k \in \{0,1\}, & \forall k \in K
\end{aligned}
$$

Application II: Transportation problem with conflicts

- Used settings from [Vancroonenburg, Della Croce, Goossens, Spieksma, (2014)]
- Number of nodes in bipartite graph: $\{50, 100\}$
- Number of transportation services: $\{20, 30\}$
- 10 instances for each size category
- Number of pairwise conflicts: 10% of total pairwise combinations
- Used Gurobi to solve different relaxations
- Used EC&R inequalities for up to 3 constraint aggregations
- Used separation to add most violated cuts

Application II: Transportation problem with conflicts

Node #	Service #	Solver		Forest EC&R		RLT	
		Opt. Time	Root Gap	Gap	Time	Gap	Time
50	20	70.74	0.1	0.53	46.01	0.65	37.61
50	30	341.2	0.19	0.54	82.77	0.69	103.13
100	20	1981.57	0.1	0.44	333.6	0.41	1590.55
100	30	≥ 5000	0.18	0.49	1164.84	0.46	2037.06

- The table shows average results over 10 instances for each size category (row)

Conclusion

Summary:

- Developed a convexfication method for bilinear set S to obtain facet-defining inequalities in the original space of variables.
- Showed that for $m = 1$, these inequalities correspond to tree structures in the underlying graph.
- Showed that for case with $m > 1$, these inequalities correspond to special forest structures in the underlying graph.
- Presented computational results to show the effectiveness of the developed methods in improving dual bounds.

Reference:

Khademnia E. and D. D. (2024) *Convexification of bilinear terms over network polytopes*. **Mathematics of Operations Research**.